sive in computing time. This is, however, something of an open question as so few structures permit alternative, analytic solutions beyond first-order perturbation.

This method is therefore proffered as a versatile and automatic procedure for analyzing, with moderate accuracy, this class of waveguide problems.

Acknowledgment

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The Green's Dyadic for Radiation in a Bounded Simple Moving Medium

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***Abstract*—The studies here show that the wave equation for electromagnetic wave propagation in an isotropic and uniformly moving medium is solvable by the separation method in four coordinate systems. Solutions in the form of complete sets of eigenfunctions are possible for problems where boundary surfaces are presented. A Green's dyadic for finite or semi-infinite domain problems involving sources in the moving medium has been formulated through vector operation on the eigenfunction solutions of the homogeneous wave equation. The case of electromagnetic waves excited by a current loop, immersed in a moving medium, and confined by a circular cylindrical waveguide, was examined. The electric and magnetic field intensities in such a waveguide were compared with those obtained through a different approach. The Green's dyadic for electromagnetic waves in an infinite domain moving medium was shown to be obtainable from the finite domain Green's dyadic through a limiting process.**

Introduction

THE PROBLEM OF electromagnetic wave propagation in a moving medium has gained a renewed interest in recent years. A number of studies has been reported on the subject involving a bounded or an unbounded moving medium. For radiation problems, Lee and Papas[1] have derived a Green's function which is adequate for sources in an infinite domain moving medium. Compton and Tai[2] also have derived an infinite domain Green's dyadic for sources in a moving medium which has a different form from that obtained by Lee and Papas. In principle, the infinite domain Green's function can be used to obtain the field in a finite domain if one retains the surface integral in the integral representation of the field. In practice, however, evaluation of the surface integral is not a simple task. For most boundary value problems involving sources inside the boundaries, the boundary conditions are usually either homogeneous Dirichlet or homogeneous Neumann, and seldom involve both homogeneous Dirichlet and homogeneous Neumann simultaneously on the same boundary surface. Any inhomogeneous boundary condition requires a priori knowledge of the surface charge density or surface current density before the surface integral can be evaluated. Such knowledge is usually not given in the statements of the problem.

To avoid such difficulties, a different approach is suggested in this paper. A study to better understand the finite or semi-infinite domain free-wave solutions is carried out.

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The finite or semi-infinite Green's dyadic is then constructed from the appropriate free-wave solutions so as to render it the same homogeneous boundary conditions the free-wave satisfied. Such homogeneous boundary conditions imposed upon the Green's dyadic facilitates the vanishing of the surface integral in the integral representation of the field.

Solution by Separation Method

In the fixed frame of reference, the Maxwell's equations and the constitutive relation for an isotropic and moving medium are

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{1}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}, \tag{2}$$

$$\nabla \cdot D = \rho, \tag{3}$$

$$\nabla \cdot B = 0, \tag{4}$$

$$D + \frac{V}{C} \times \left(\frac{V}{C} \times D\right) = \epsilon E + \epsilon \frac{V}{C_0} \times \left(\frac{V}{C_0} \times E\right) + \left(\frac{1}{C^2} - \frac{1}{C_0^2}\right) V \times H, \tag{5}$$

$$B + \frac{V}{C} \times \left(\frac{V}{C} \times B\right) = \mu H + \mu \frac{V}{C_0} \times \left(\frac{V}{C_0} \times H\right) - \left[\frac{1}{C^2} - \frac{1}{C_0^2}\right] V \times E, \tag{6}$$

where V is the velocity of the moving medium, μ and ϵ are the permeability and the permittivity, respectively, of the medium at rest, $C = 1/\sqrt{\mu\epsilon}$, and C_0 is the velocity of light in free space. Here, it is assumed that J and ρ are the source current density function and the source charge density function, respectively, expressed in the fixed frame coordinates.

Assuming harmonic variation of the form $e^{j\omega t}$, manipulation of (1) to (6) yields the wave equation for E and H, respectively:

$$\overline{L} \times \overline{L} \times E - k^2 E = -j\omega\mu \overline{\alpha}^{-1} \cdot J, \tag{7}$$

$$\overline{L} \times \overline{L} \times H - k^2 H = \overline{L} \times \overline{\alpha}^{-1} \cdot J. \tag{8}$$

Two additional equations corresponding to (3) and (4) may also be written

$$\overline{\alpha} \cdot \overline{L} \cdot (\epsilon \overline{\alpha} \cdot E) = \rho - \Omega \cdot J, \tag{9}$$

$$\overline{\alpha} \cdot \overline{L} \cdot (\mu \overline{\alpha} \cdot H) = 0, \tag{10}$$

where

$$k^2 = \omega^2 \mu\epsilon, \qquad \overline{L} = \overline{\alpha}^{-1} \cdot (\nabla - i\omega\Omega).$$

$$\overline{\alpha} = \begin{bmatrix} 1 - \frac{V^2}{C_0^2} + \frac{(V\cdot a_1)^2}{C_0} & \frac{(V\cdot a_1)(V\cdot a_2)}{C_0^2} & \frac{(V\cdot a_1)(V\cdot a_3)}{C_0^2} \\ \frac{(V\cdot a_2)(V\cdot a_1)}{C_0^2} & 1 - \frac{V^2}{C_0^2} + \frac{(V\cdot a_2)^2}{C_0^2} & \frac{(V\cdot a_2)(V\cdot a_3)}{C_0^2} \\ \frac{(V\cdot a_3)(V\cdot a_1)}{C_0^2} & \frac{(V\cdot a_3)(V\cdot a_2)}{C_0^2} & 1 - \frac{V^2}{C_0^2} + \frac{(V\cdot a_3)^2}{C_0^2} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_1)^2}{C^2} & \frac{(V\cdot a_1)(V\cdot a_2)}{C^2} & \frac{(V\cdot a_1)(V\cdot a_3)}{C^2} \\ \frac{(V\cdot a_2)(V\cdot a_1)}{C^2} & 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_2)^2}{C^2} & \frac{(V\cdot a_2)(V\cdot a_3)}{C^2} \\ \frac{(V\cdot a_1)(V\cdot a_3)}{C^2} & \frac{(V\cdot a_3)(V\cdot a_2)}{C^2} & 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_3)^2}{C^2} \end{bmatrix}^{-1},$$

$$\Omega = \left[\frac{1}{C^2} - \frac{1}{C_0^2}\right] \begin{bmatrix} 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_1)}{C^2} & \frac{(V\cdot a_1)(V\cdot a_2)}{C^2} & \frac{(V\cdot a_1)(V\cdot a_3)}{C^2} \\ \frac{(V\cdot a_2)(V\cdot a_1)}{C^2} & 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_2)^2}{C^2} & \frac{(V\cdot a_2)(V\cdot a_3)}{C^2} \\ \frac{(V\cdot a_1)(V\cdot a_3)}{C^2} & \frac{(V\cdot a_3)(V\cdot a_2)}{C^2} & 1 - \frac{V^2}{C^2} + \frac{(V\cdot a_3)}{C^2} \end{bmatrix}^{-1} \cdot V,$$

and $\bar{\alpha}\cdot\bar{\alpha}^{-1}=\bar{\alpha}^{-1}\cdot\bar{\alpha}=\bar{I}$, with $\bar{I}$ being the idemfactor. In a source-free region, (7) to (10) reduces to homogeneous equations of the form:

$$\bar{L}\times\bar{L}\times F - k^2F = 0, \tag{11}$$

and

$$\bar{\alpha}\cdot\bar{L}\cdot(\bar{\alpha}\cdot F) = 0, \tag{12}$$

where F represents E or H. Equation (11) is seen to resemble a vector Helmholtz equation except that the operator $\bar{L}$ involves an additional term and a multiplying dyadic. It is well known that the scalar Helmholtz equation is solvable by separation method in eleven coordinate systems, and that the vector Helmholtz equation is separable in only six coordinate systems. As the Helmholtz equation takes on more complicated form, it is anticipated that the number of coordinate systems in which it is separable shall become less. Despite the fact that (11) has been solved for some problems mostly involving the rectangular coordinate system and the circular cylindrical coordinate system, its separability has not been seriously studied. Such study is desirable since by determining the coordinate systems in which this equation is separable, one not only has the knowledge of exactly in what coordinate system one may solve this equation by a separation method, but one also may attempt solutions in the form of eigenfunction when boundaries are present. The eigenfunction solution will be of great help in constructing the finite domain or the semi-infinite domain Green's dyadic.

The first term in (11), after expansion, contains a vector operation term in the form of (curl curl F). A review of the separability of a vector Helmholtz equation shows that the coordinate systems in which the (curl curl F) term facilitates separation must be a coordinate system in which one of the scale factors is unity and that the ratio of the remaining two scale factors must be independent of the coordinate corresponding to the unity scale factor.[3] The six coordinate systems which meet these requirements are the conical, the spherical, and the four cylindrical coordinate systems.

The constitutive relation (5) and (6) were obtained through a proper Lorentz transformation, the velocity V of the material medium must be constant in magnitude as well as in direction. Of the six coordinate systems which permit separation of the term (curl curl F), only the four cylindrical coordinate systems permit simple expression of constant direction in V. In fact, the most simple expression of V is obtained by orienting the chosen coordinate system such that the unity scale factor coordinate be parallel or antiparallel to V. Without losing generality, let this coordinate be denoted ξ_3, and its unit vector denoted a_3. V is then simply expressed as Va_3, where V is the magnitude of V. In Cartesian coordinate systems, ξ_3 may represent X, Y, or Z. In the remaining three cylindrical coordinate systems, i.e., circular, elliptical, and parabolic, ξ_3 represents the Z coordinate only. Using $V=Va_3$, the expressions for Ω and $\bar{\alpha}$ are then reduced to those obtained by Tai,[4]

$$\Omega = \frac{(n^2-1)\beta}{(1-n^2\beta^2)C}\,a_3 = \Omega a_3, \tag{13}$$

$$\bar{\alpha} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where

$$n = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}, \qquad \beta = \frac{V}{C_0}, \qquad \alpha = \frac{1-\beta^2}{1-n^2\beta^2}. \tag{15}$$

In the application of boundary-value problems, separation into the form that conveniences the fitting of boundary surfaces is most desirable. It is suggested that (11) and (12) be separated in terms of the two transverse vectors M and N,

$$F = M + N, \tag{16}$$

$$M = \bar{L}\times\phi a_3 = -\frac{1}{\alpha}\,a_3\times\nabla_\perp\phi, \tag{17a}$$

$$N = \bar{L}\times\bar{L}\times\chi a_3 = -\frac{1}{\alpha}(\nabla_\perp^2\phi)a_3 + \frac{1}{\alpha^2}\nabla_\perp(\nabla_\parallel\chi - j\omega\Omega\chi), \tag{17b}$$

where ϕ and χ are scalar functions to be determined. The subscript $\perp$ indicates the components of operator or vector which are perpendicular to ξ_3, whereas $\parallel$ indicates parallel to ξ_3. The vector M is seen to be entirely tangential to the surface $\xi_3=$constant. The vector N, in general, has a component tangential to the $\xi_3=$constant surface and a component normal to it, thus tangential to the $\xi_1=$constant and $\xi_2=$constant surfaces.

By invoking the identity $\bar{M}\cdot(\bar{M}\times A)=0$, where A is any vector function and $\bar{M}=\bar{\alpha}\cdot\bar{L}$, it can be shown that both M and N satisfy (12). Substitution of (16) and (17), respectively, into (11) yields the scalar differential equation:

$$\left(\nabla_\perp^2 + \frac{1}{\alpha}\nabla_\parallel^2 + \frac{j2\omega\Omega}{\alpha}\,a_3\cdot\nabla_\parallel - \frac{\omega^2\Omega^2}{\alpha} + k^2\alpha\right)f = 0, \tag{18}$$

where f represents either ϕ or χ. It is evident that if (18) can be solved to the satisfaction of the appropriate boundary conditions, solution for E can be obtained through (16) and (17). Equation (18) is solved easily by separation of variables method, which demands

$$\nabla_\perp^2 f = -K_{m,n}^2 f, \tag{19}$$

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[4] C. T. Tai, "The dyadic Green's function for a moving isotropic medium," *IEEE Trans. Antennas and Propagation* (*Communications*), vol. AP-13, pp. 322–323, March 1965.

and

$$a_3 \cdot \nabla_\| f = \pm jk_{mn} f, \tag{20}$$

where K_{mn} and k_{mn} are the separation constant, they are related by the dispersion relation

$$K_{mn}^2 + \frac{1}{\alpha} k_{mn}^2 \pm \frac{2\omega\Omega}{\alpha} k_{mn} + \frac{\omega^2\Omega^2}{\alpha} - k\alpha^2 = 0. \tag{21}$$

The subscript m, n is required to signify that either K_{mn} or k_{mn} or both are eigenvalues depending upon the manner in which the boundaries set up in the problem. In view of (19) and (20), the solution for f should take the form of eigenfunctions

$$f = \sum_{m,n} f_{\perp mn}(\xi_1, \xi_2, K_{mn}) f_{\| mn}(\xi_3, k_{mn}). \tag{22}$$

If the boundaries are parallel to the ξ_3=constant surface, $f_{\|_{mn}}$, then $\chi_{\|_{mn}}(\xi_3)$ or $\phi_{\|_{mn}}(\xi_3)$ are sets of eigenfunctions, k_{mn} are the eigenvalues running on one index, say m, and at least a component of K_{mn} obtained from the dispersion relation (21) will describe the dispersion relation for propagation in the (ξ_1, ξ_2) space. The remaining component of K_{mn} will be another eigenvalue running on the index n, with $\phi_{\perp_{mn}}(\xi_1, \xi_2)$ and $\chi_{\perp_{mn}}(\xi_1, \xi_2)$ describing propagation in the (ξ_1, ξ_2) space. Conversely, if the boundary surface is perpendicular to the ξ_3=constant surfaces, $\phi_{\perp_{mn}}(\xi_1, \xi_2)$ and $\chi_{\|_{mn}}(\xi_3)$ will consist of sets of eigenfunctions with K_{mn} describing the eigenvalues. $\phi_{\|_{mn}}(\xi_3)$ and $\chi_{\|_{mn}}(\xi_3)$ describe the propagation in the ξ_3 direction with k_{mn} being the parameter describing the dispersion relation. Of course, if the boundary is a self-enclosed one, the two sets of indices, m and n, will degenerate into three sets, m, n, and 1.

The solution f from (22), therefore ϕ and χ, is complete if $f_{\perp m}$ and $f_{\|_{mn}}$ as obtained from (19) and (20) each constitutes a complete set when sum on all indices; consequently, M and N as well as F or E and H, constructed from the complete set of ϕ and χ, must be complete. Furthermore, it can be shown that for a physically realizable problem, the functional forms of $\phi_{\perp_{mn}}$, $\chi_{\perp_{mn}}$, $\phi_{\|_{mn}}$, and $\chi_{\|_{mn}}$ are so chosen as to satisfy the boundary conditions for E and H, the solutions of E and H constructed from ϕ and χ must be unique.

Inhomogeneous Equation and the Green's Dyadic

When sources are presented in the bounded region, (7) and (8) can be represented by a symbolic equation

$$\bar{L} \times \bar{L} \times F - k^2 F = J_s(r), \tag{23a}$$

where

$$J_s(r) = -j\omega\mu\bar{\alpha}^{-1} \cdot J(r), \tag{23b}$$

if F represents E, and

$$J_s(r) = \bar{L} \times \bar{\alpha}^{-1} \cdot J(r), \tag{23c}$$

if F represents H.

The integral representation of F in (23a) is sought,

$$F = \int_{v_0} J_s(r_0) \cdot \bar{G}(r \mid r_0) dv_0, \tag{24}$$

where the kernel $\bar{G}(r|r_0)$ is the Green's dyadic which satisfies

$$\bar{L} \times \bar{L} \times \bar{G}(r \mid r_0| - k^2 \bar{G}(r \mid r_0) = -4\pi \bar{I}\delta(r - r_0), \tag{25}$$

where the vector operator $\bar{L}$ is on the observer coordinate r. It is also required that $\bar{G}(r|r_0)$ must satisfy

$$\bar{L}_0^* \times \bar{L}_0^* \times \bar{G}(r \mid r_0) - k^2 \bar{G}(r \mid r_0) = -4\pi \bar{I}\delta(r - r_0), \tag{26}$$

where the subscript 0 on the vector operator indicates that the operation is on source coordinates and the asterisk denotes complex conjugate, thus

$$\bar{L} = \bar{\alpha}^{-1} \cdot (\nabla - j\omega\Omega), \tag{27a}$$

$$\bar{L}_0^* = \bar{\alpha}^{-1} \cdot (\nabla_0 + j\omega\Omega). \tag{27b}$$

The requirement that the complex conjugate of the vector operator be used while interchange the observer and source coordinates is the best direct consequence of the reciprocity theorem for the Green's dyadic.[5] The $j\omega$ term in (27) actually comes from time operator $\partial/\partial t$ operating on the harmonic factor $e^{j\omega t}$. The reciprocity of the Green's dyadic for an equation involving both spatial and temporal operations implies that

$$\bar{G}(r, t \mid r_0, t_0) = \bar{G}(r_0, -t_0 \mid r, -t). \tag{28}$$

The need to use the complex conjugate vector operator $\bar{L}_0$ in (26) is therefore evidenced.

To obtain (24), one rewrites (23a) in source coordinates and manipulates it with (26) to obtain

$$F(r) = \frac{1}{4\pi}\int_{v_0} \bar{J}_s(r_0) \cdot \bar{G}(r \mid r_0) dv_0 + \frac{1}{4\pi}\int_{v_0} [F(r_0) \cdot \bar{L}_0^* \times \bar{L}_0 \times \bar{G} - (\bar{L}_0 \times \bar{L}_0 \times F(r_0)) \cdot \bar{G}] dv_0. \tag{29}$$

The second volume integral to the right of the equality in (29) can be changed into a surface integral by the identity

$$A \cdot \bar{L}^* \times \bar{L}^* \times B - (\bar{L} \times \bar{L} \times A) \cdot B = \nabla \cdot [A \times \bar{L}^* \times B - (\bar{L} \times A) \times B]. \tag{30}$$

Thus

$$F(r) = \frac{1}{4\pi}\int_{v_0} \bar{J}_s \cdot \bar{G} dv_0 + \frac{1}{4\pi}\oint [F \times \bar{L}^* \times \bar{G} - (\bar{L}_0 \times F \times \bar{G})] \cdot dS_0. \tag{31}$$

If the surface integral in (31) vanishes, then (24) will result. Since the boundary condition for F is usually homogeneous, regardless of whether F satisfies a homogeneous Dirichlet boundary condition or a homogeneous Neumann's boundary condition, the surface integral in (31) will vanish if it is demanded that the Green's dyadic satisfies the same boundary condition that F satisfies. Such condition imposed upon

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the Green's dyadic can easily be fulfilled if the Green's dyadic is constructed from the solution of F for the homogeneous equation as given in (16) and (17), providing that F so obtained satisfies the appropriate boundary condition. In view of the preceding arguments and the fact that $\bar{G}(r|r_0)$ must satisfy both (25) and (26), it is obvious that $\bar{G}(r|r_0)$ should take the form

$$\bar{G}(r|r_0) = \sum_{m,n}\left[\frac{1}{{}_1\Lambda_{mn}}\bar{L}\times\phi(r)a_3 + \frac{1}{{}_2\Lambda_{mn}}a_3\nabla_\perp^2\chi(r) + \frac{1}{{}_3\Lambda_{mn}}\nabla_\perp(a_3\cdot\nabla_\|\chi_{mn}(r)+j\omega\Omega\chi_{mn}(r))\right]$$
$$[\bar{L}_0\times\phi_{mn}^*(r_0)a_3 + a_3\nabla_{\perp_0}^2\chi_{mn}^*(r_0) + \nabla_{\perp_0}(a_3\cdot\nabla_{\|_0}\chi_{mn}^*(r_0) + j\omega\Omega\chi_{mn}^*(r_0))] \quad (32)$$

where $\phi_{mn}(r)$ and $\chi_{mn}(r)$ must satisfy (18) or (19) and (20) simultaneously, as well as satisfying the boundary conditions for F. $\phi_{mn}^*(r_0)$ and $\chi_{mn}^*(r_0)$ are the complex conjugate of $\phi_{mn}(r)$ and $\chi_{mn}(r)$, respectively, and are expressed in the source coordinates. The constants $1/\Lambda_{mn}$ can be evaluated by substituting (32) into (25), which yields

$$\sum_{m,n}\left(-K_{mn}^2 - \frac{k_{mn}^2}{\alpha} \pm \frac{2\omega\Omega k_{mn}}{\alpha} - \frac{\omega^2\Omega^2}{\alpha} + k\alpha^2\right)[\bar{G}(r|r_0)] = -4\pi\bar{I}\delta(r-r_0). \quad (33)$$

Although $\bar{L}\times\phi a_3$ and $\bar{L}\times\bar{L}\times\chi a_3$ are not mutually perpendicular in space, but their components $a_3\times\nabla_\perp\phi$, $a_3\nabla_\perp^2\chi$, and $\nabla_\perp(a_3\cdot\nabla_\|\chi - j\omega\Omega\chi)$ are three mutually orthogonal vectors. Multiply, in turn, $a_3\times\nabla_\perp\phi^*$, $a_3\nabla_\perp^2\chi^*$, and $\nabla_\perp(a_3\cdot\nabla_\|\chi^* + j\omega\Omega\chi^*)$ scalarly from the left into both sides of the equality sign in (33). Use the mutually orthogonal vector properties, the resulting equation can be written into three equations

$$\sum_{mn}\frac{1}{\alpha^6\Lambda_1}Z_{mn}(a_3\times\nabla_\perp\phi_{pq}^*(r))\cdot(a_3\times\nabla_\perp\phi_{mn}(r))(a_3\times\nabla_{\perp_0}\phi_{mn}(r)) = \frac{-4\pi}{\alpha}(a_3\times\nabla_\perp\phi_{pq}^*(r))\delta(r-r_0),$$

$$\sum_{mn}\frac{1}{\alpha^3\Lambda_2}Z_{mn}(\nabla_\perp^2\chi_{rs}^*(r)a_3)\cdot(\nabla_\perp^2\chi_{mn}(r)a_3)(\nabla_{\perp_0}^2\chi_{mn}^*(r_0)a_3) = \frac{-4\pi}{\alpha}\nabla_\perp^2\chi_{rs}^*(r)a_3\delta(r-r_0), \quad (34)$$

$$\sum_{mn}\frac{1}{\alpha^6\Lambda_3}Z_{mn}\nabla_\perp\left(\frac{\partial\chi^*_{rs}(r)}{\partial z} + j\omega\Omega\chi_{rs}^*(r)\right)\cdot\nabla_\perp\left(\frac{\partial\chi_{mn}(r)}{\partial z} - j\omega\Omega\chi_{mn}(r)\right)\nabla_{\perp_0}\left(\frac{\partial\chi^*}{\partial z} + j\omega\Omega\chi_{mn}^*\right) = \frac{-4\pi_{rs}}{\alpha^2}\nabla_\perp\left(\frac{\partial\chi_{rs}}{\partial z} + j\omega\Omega\chi_{rs}\right)\delta,$$

where

$$Z_{mn} = -K_{mn}^2 - \frac{k_{mn}}{\alpha} \pm \frac{2\omega\Omega k_{mn}}{\alpha} - \frac{\omega^2\Omega^2}{\alpha} + k^2\alpha.$$

Taking advantage of (19) and (20), (34) can be integrated over the entire observer space. Utilizing the orthogonality properties of the eigenfunction ϕ_{mn} and χ_{mn}, and the singular property of the delta function, one obtains

$${}_1\Lambda_{mn} = \frac{|K_{mn}^2|Z_{mn}N_{mn}^\phi}{\alpha^2 4\pi}, \quad (35a)$$

$${}_2\Lambda_{mn} = -\frac{|K_{mn}^4|Z_{mn}N_{mn}^\chi}{\alpha^2 4\pi}, \quad (35b)$$

$${}_3\Lambda_{mn} = \frac{(k_m+\omega\Omega)^2 Z_{mn}K_{mn}^2N_{mn}^\chi}{\alpha^4 4\pi}, \quad (35c)$$

where N_{mn}^ϕ and N_{mn}^χ are the normalization factor for ϕ_{mn} and χ_{mn}, respectively. The complete Green's function for finite or semi-infinite domain is therefore

$$G(r|r_0) = \sum_{m,n}\frac{4\pi\alpha^2}{K_{mn}^2Z_{mn}N_{mn}^\phi}(\nabla_\perp\phi_{mn}(r)\times a_3)(\nabla_{\perp_0}\phi_{mn}^*(r)\times a_3) - \sum_{m,n}\frac{4\pi\alpha^2}{K_{mn}^4Z_{mn}N_{mn}^\chi}\nabla_\perp^2\chi_{mn}(r)a_3a_3\nabla_{\perp_0}^2\chi_{mn}^*(r_0)$$
$$+\sum_{m,n}\frac{4\pi\alpha^2}{K_{mn}^2(k_{mn}+\omega\Omega)^2Z_{mn}N_{mn}^\chi}\nabla_\perp(\nabla_\|\cdot a_3\chi_{mn}(r) - j\omega\Omega\chi_{mn})\nabla_{\perp_0}(\nabla_{\|0}\cdot a_3\chi_{mn}^*(r) + j\omega\Omega\chi_{mn}^*(r_0)). \quad (36)$$

Symbolically, (36) may be written as

$$G(r|r_0) = \sum^{i}\sum^{m,n}(L^i)(L_0^i)^*S_{mn}^i g_{mn}^i(r|r_0), \quad (37)$$

where (L^i) and $(L_0^i)^*$ are the vector operators in (36); S_{mn}^i are the constants in (36) involving N_{mn}'s, K's, k's, and α. g_{mn}^i is a scalar Green's function such that

$$g_{mn}^i = \phi_{mn}(r)\phi_{mn}^*(r_0),$$

if i runs on the term involving the operators $(L^i) = (-a_3\times\nabla_\perp)$ and $(L_0^i)^* = (-a_3\times\nabla_{\perp_0})$. And

$$g_{mn}^i = \chi_{mn}(r)\chi_{mn}(r_0),$$

if i runs on the term involving the operators $(L^i) = a_3\nabla_\perp^2$, $(L_0^i)^* = a_3\nabla_{\perp0}^2$, or $(L^i) = \nabla_\perp(a_3\cdot\nabla_\| - j\omega\Omega)$ and $(L_0^i) = \nabla_{\perp0}(a_3\cdot\nabla_{\|0} + j\omega\Omega)$.

It is obvious that this scalar Green's function satisfies the scalar Green's equation

$$\left(\nabla_\perp^2 + \frac{1}{\alpha}\nabla_\|^2 - \frac{2i\omega\Omega}{\alpha}a_3\cdot\nabla_\| - \frac{\omega^2\Omega^2}{\alpha} + k^2\alpha\right)g_{mn}^i = \delta(r-r_0). \quad (38)$$

The symbolical form of $G(r|r_0)$ as given in (37) would be of convenience for the discussion of retrieving the infinite domain Green's function to be presented after the next section.

Excitation in a Circular Cylindrical Waveguide

As an example, the case of a circular cylindrical waveguide with perfectly conducting wall and filled with a moving dielectric medium (μ, ϵ) having a velocity V along the longitudinal axis of the guide is examined here. The guide is assumed to have a radius a and is oriented with its longitudinal axis along z axis so that $V = Va_z$. The wave is assumed to be excited by an infinitesimally thin current loop $I_0 e^{j\omega t}$ located at the radius $r = a/2$ and the plane $z = 0$, as shown in Fig. 1.

The source current density is expressed as

$$J_s = \frac{I_0 \delta(r - a/2)\delta(z)}{2\pi r} e^{j\omega t} a_\theta. \tag{39}$$

One may start with (7) for the electric field intensity E; using the source function for E as given by (23b)

$$J_s = -j\omega\mu\alpha a_\theta \left(\frac{I_0 \delta(r - a/2)\delta(z)}{2\pi r} \right). \tag{40}$$

The appropriate free-wave solutions of ϕ and χ that satisfy (18), (19), and (20), and also satisfy boundary conditions at the wall can be written immediately

$$\phi_{mn} = J_n(K_{mn}{}^\phi r) e^{jn\phi} e^{ik_{mn}{}^\phi z} e^{j\omega t}, \tag{41a}$$

$$\chi_{pq} = J_q(K_{pq}{}^\chi r) e^{jq\theta} e^{ik_{pq}{}^\chi z} e^{j\omega t} \tag{41b}$$

where m, n, p, and q are all integers. $(K_{mn}{}^\phi a)$ is the mth zero of the first derivative of the nth order Bessel function $(\partial/\partial r) J_n(K^\phi r)$, $(K_{pq}{}^\chi a)$ is the pth zero of the qth order Bessel function $J_q(K_{pq} r) \cdot k_{mn}{}^\phi$ and $k_{pq}{}^\chi$ are the propagations constants for the ϕ_{mn} wave mode and χ_{mn} wave mode, respectively. They are evaluated from the dispertion relations (21); each yields four roots

$$k_{mn}{}^\phi = \pm \left[\omega\Omega \pm \sqrt{k^2\alpha^2 - K_{mn}{}^\phi \alpha} \right], \tag{42a}$$

$$k_{mn}{}^\chi = \pm \left[\omega\Omega \pm \sqrt{k\alpha^2 - K_{mn}{}^\chi \alpha} \right]. \tag{42b}$$

The signs outside the square bracket are chosen such that they represent wave traveling away from $z = 0$ plane.

Equation (42) is essentially that obtained by Du and Compton,[6] except that Du and Compton only consider the two roots with minus sign outside the bracket. It is of interest to note that (41a) yields solutions corresponding to the TE mode waves in Collier and Tai's analysis[7] while (41b) corresponds to their TM mode waves. The complete free-wave solution for E is therefore

$$E = \sum_{mn} A_{mn} \{ \nabla_\perp (J_n(K_{mn}{}^\phi r) e^{in\theta}) \times a_3 \} e^{ik_{mn}{}^\phi z} + \sum_{p,q} B_{pq} \overline{L} \times \overline{L} \times [J_{pq}(K_{pq}{}^\chi r) e^{iq\theta} e^{ik_{pq}{}^\chi z}] a_3, \tag{43}$$

[6] J. L. Du and R. T. Compton, Jr., "Cutoff phenomena for guided waves in moving media," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-14, pp. 358–363, August 1966.

[7] J. R. Collier and C. T. Tai, "Guided waves in moving media," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-13, pp. 441–445, July 1965.

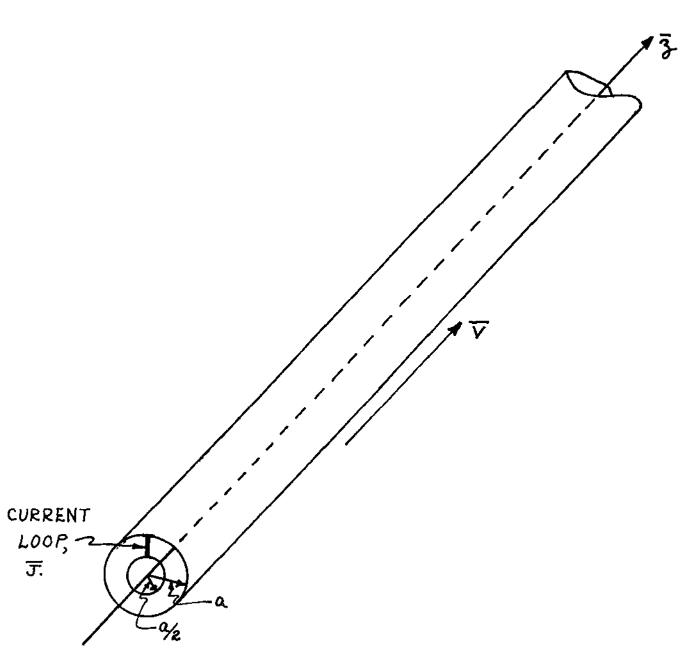

Fig. 1. Cylindrical waveguide with loop current source.

where A_{mn} and B_{mn} are arbitrary constants. The magnetic field strength can be obtained from E through the modified Maxwell's equation

$$\overline{L} \times E = -j\omega\mu H. \tag{44}$$

The Green's function for E in this waveguide is constructed from ϕ_{mn} and χ_{pq} according to (36)

$$\begin{aligned} G(r \mid r_0) = {} & \sum_{m,n} \frac{4\pi\alpha^2}{K_{mn}{}^2 Z_{mn} N_{mn}{}^\phi} (\nabla_\perp \phi_{mn}(r) \times a_3)(\nabla_{\perp_0} \phi_{mn}(r) \times a_3) \\ & - \sum_{m,n} \frac{4\pi\alpha^2}{K_{mn}{}^4 Z_{mn} N_{mn}{}^\chi} \nabla_\perp{}^2 \chi_{mn}(r) a_3 a_3 \\ & + \sum_{m,n} \frac{4\pi\alpha^2}{K_{mn}{}^2 (k_m + \omega\Omega)^2 Z_{mn} N_{mn}{}^\chi} \nabla_\perp (\nabla_\parallel \cdot a_3 \chi_{mn}(r) - j\omega\Omega\chi_{mn}) \nabla_{\perp_0} \\ & (\nabla_\parallel \cdot a_3 \chi_{mn}{}^*(r) + j\omega\Omega\chi_{mn}{}^*(r_0)). \end{aligned} \tag{45}$$

The field excited by the current loop is found by substituting (45) and (40) into (24) and perform the integral. The integration is not difficult, if one observes that the source function is a transverse function, only the transverse term of the Green's function is needed. Furthermore, $J(r)$ does not depend on θ by reason of orthogonality of the eigenfunctions, there shall be no contribution from $n \leq 1$ terms in the Green's function, one obtains:

$$E(r) = a_\theta \frac{j\omega\mu\alpha I_0}{8\pi^2} A_{01} J_0{}' \left(\frac{K_{01}}{2} \right) J_0{}'(K_{01} r) e^{jk_{01} z}, \tag{46a}$$

where $(K_{01}a)$ is the first zero of J_0', the prime on the Bessel functions denotes the first derivative of the Bessel function with respect to $(K_{01}r)$. Using (45)

$$H = \frac{-\alpha_0 I_0}{8\pi^2} A_{01} J_0'\left(\frac{K_{01}}{2}\right) \bar{L} \times (J_0'(K_{01}r)e^{jk_{01}z})a_\theta, \qquad (46b)$$

where

$$A_{01} = \frac{4\pi\alpha^2}{K_{01}^2 Z_{01} N_{01}^{\phi}}, \qquad (47a)$$

$$N_{01} = \frac{\pi}{4}. \qquad (47b)$$

In view of (45), it is seen that the current loop excites a TE_{01} mode wave in the circular cylindrical waveguide.

The Infinite Domain Green's Dyadic

It is interesting to note that the transition of a finite or semi-infinite domain Green's dyadic may be obtained through a limiting process. No attempt will be made here to derive the infinite domain Green's dyadic into its final form. The main purpose here is to show that such transition is possible.

As an example, take the Green's dyadic of the circular cylindrical waveguide given in (45). As the wall recedes to infinity, i.e., $a \to \infty$, the summation on m goes over to an integral. Written in the symbolic form of (38), the transformed infinite domain Green's dyadic, $G_\infty(r \mid r_0)$, is

$$G_\infty(r \mid r_0) = \sum_i \sum_n (L^i)(L^{j*}) \int_0^\infty S^i(k) g_n{}^i dk. \qquad (48)$$

In (48), the order of integration and differentiation operations has been interchanged and the subscript m has been dropped. Evidently, the Green's dyadic for infinite domain can be obtained through a set of auxiliary scalar functions, I^i, as represented by the integral in (48). In view of the fact that $g_n{}^i$ involves $e^{jn\theta}$ and the Bessel functions, after using the addition theorem to perform the summation on n, I^i may be written

$$I^i = \frac{1}{2} \int_{-\infty}^{+\infty} S^i H_0^{(2)}(Kr_\perp') e^{-jkz'} dk, \qquad (49)$$

where

$$r_\perp' = \left| r_\perp - r_{\perp 0} \right|, \qquad z' = \left| z - z_0 \right|,$$

and $H_0^{(2)}(Kr_\perp')$ is the Hankel function of the second kind of the zero order. It should be noted that the new coordinate system has its origin at the source point. This choice of a new origin may require subsequent transformation back to the original origin.

An asymptotic solution which is valid for waves at large distance from the source will be attempted through the method of steepest descent.

Assuming that interest is in the accuracy of the solution only to the order of $r_\perp'$, the zero order Hankel function may be expanded into its asymptotic form. Retaining only the first term, (49) then becomes

$$I^i = \frac{e^{-j(\pi/4)}}{2} \int_{-\infty}^{+\infty} \frac{S^i e^{-jkz'} e^{-jkr_\perp'}}{(\pi K r_\perp')^{1/2}} dk. \qquad (50)$$

At this point, it would seem more convenient to change the coordinate system from that of circular cylindrical coordinates to that of spherical coordinates (R, ϕ, α); where

$$r_\perp' = R \sin \phi,$$
$$z' = R \cos \phi.$$

Under the new coordinate system, (50) becomes

$$I^i(R) = \frac{e^{-j(\pi/4)}}{2} \int_{-\infty}^{+\infty} \frac{S^i}{(\pi K R \sin \phi)^{1/2}} \cdot e^{-jR(K \sin \phi + k \cos \phi)} dk. \qquad (51)$$

It is recalled that K and k are related through the dispersion relations (21). Define a new parameter

$$T^2 = K^2 + k^2 \qquad (52)$$

T is therefore the total propagation factor. Now for the sake of convenience, instead of k, a new integration parameter, τ, may be employed, such as

$$K = T \sin \tau, \qquad (53a)$$
$$k = T \cos \tau. \qquad (53b)$$

The parameter τ has the same significance as the angle which measures the wave normal if T is a constant; however, in the present case, T is not a constant. In fact, combining (21), (52), and (53) yields an expression for T in terms of τ.

$$T_{1 \text{ to } 4} = \frac{\pm \frac{\omega\Omega}{\alpha} \cos \tau \pm \sqrt{k^2(\cos^2 \tau + \alpha \sin^2 \tau) - \frac{\omega^2\Omega^2}{\alpha} \sin^2 \tau}}{\sin^2 \tau + 1/\alpha \cos^2 \tau} \qquad (54)$$

where the subscript 1 to 4 on T represents the choice of plus or minus sign in (54). For simplicity, the subscripts on T are dropped. The integral for I^i becomes

$$I^i = \frac{1}{(j4R \sin \phi)^{1/2}} \int_C \frac{S^i}{[\pi T(\tau) \sin \tau]^{1/2}} e^{-Ru(\tau)} d\tau, \qquad (55)$$

where

$$u(\tau) = jT(\tau) \cos (T - \phi). \qquad (56)$$

Examination of the exponent shows that the real part of $u(\tau)$ approaches $+\infty$ as k approaches $\pm\infty$. The saddle point of the integration is determined by

$$\frac{du}{d\tau} = 0, \qquad (57)$$

which yields

$$\frac{1}{T(\tau_0)}\left(\frac{d}{d\tau}T(\tau)\right)_{\tau=\tau_0} = \tan(\tau_0 - \phi). \quad (58)$$

The contour of integration, C, is then chosen such that the path goes through the saddle point, τ_0, and that the imaginary part of u is constant. Following the method of steepest descent, the solution for (55) is therefore

$$I^i = \frac{S^i(\tau_0)}{2n(\tau_0)} \frac{e^{-Ru(\tau_0)}}{R(\sin\phi)^{1/2}} \quad (59)$$

where

$$n(\tau_0) = \left\{2T\sin\tau\left[\left(\frac{dT}{d\tau} - T\right)\cos(\tau-\phi) - 2\frac{dT}{d\tau}\sin(\tau-\phi)\right]\right\}^{1/2}_{\tau=\tau_0}. \quad (60)$$

The electric field intensity $\boldsymbol{E}$ in infinite domain may, therefore, be obtained from

$$\boldsymbol{E} = \sum_i \int_{v_0}\left[(L^i)(L_0{}^{*i})\frac{S^i(\tau_0)}{2n(\tau_0)}\frac{e^{-jRu(\tau_0)}}{R(\sin\phi)^{1/2}}\right]\cdot \boldsymbol{I}_s dv_0 \quad (61)$$

providing that all parameters, including the differential operators, are properly transformed to the correct observer and source coordinates in the spherical coordinate system.

The Green's dyadic as shown in (61) is in a form different from those obtained either by Lee and Papas or by Compton and Tai. Nevertheless, the results should be equivalent.

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Coupler-Type Bend for Pillbox Antennas

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Abstract—A new type of 180° *H*-plane bend has been developed for use in double-layer pillbox antennas. This bend, called a coupler-type bend, permits complete coupling between two pillbox layers with a minimum of reflection, cross-polarization, and defocusing. It can be used with short focus antennas where large feed angles are involved. The coupler-type bend utilizes a metal plate between the pillbox layers; the plate contains a pattern of holes which achieves the desired coupling.

Analytical and experimental programs have been implemented to determine the optimum hole size and distribution. Simulation techniques in rectangular waveguide were employed for convenience in measurements. The bend design was measured to have a reflection less than 2 dB SWR over a ten percent frequency band; this is computed to contribute less than 0.2 dB SWR to the reflection seen by the feed-horn of a double-layer pillbox. The bend introduces less than −22 dB of cross-polarization in the antenna radiation. Measurements of a pillbox model incorporating the bend design have verified the predicted performance of the coupler-type bend.

I. Introduction

ONE OF THE common antennas for generating a fan beam is the pillbox or "cheese" antenna [1], [2]. For the single-layer pillbox antenna, the feed is usually located in the radiating aperture, producing several undesirable effects. First, a portion of the wave in the aperture is received by the feed and appears as a reflection. Second, this blocking causes a hole in the aperture excitation giving rise to degradation of the pattern in the form of higher sidelobes.

In order to eliminate these effects, double-layer pillboxes have been developed [2]. In the double-layer arrangement, the feed is located in one layer with the second layer containing the aperture. In such designs, the principal problem is to transfer the wave efficiently from one layer to the other. In the past, this has been accomplished with a bend consisting of a large slot in the common wall between the layers, bordering along the entire length of the parabolic reflector. In general, this configuration has been successful only over a narrow bend of frequencies and for long focal-length pillboxes, in which the bend is required to operate only over a narrow range of angles. In addition, there is frequently the problem of appreciable antenna response to cross-polarization. All of these defects can be traced to the performance of the coupling device.

A new double-layer pillbox has been developed at Wheeler Laboratories which operates over a substantially larger band of frequencies and with a larger feed angle (shorter focal length) than possible in the past. Also, cross-polarization is suppressed to a tolerable level. This performance has been achieved by an improved coupling device between the layers; the device is referred to as a coupler-type bend [3].

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